AN ANALYTIC SOLUTION OF THE BECKMANN SCATTERING INTEGRAL WITH APPLICATION TO TEMPERATURE MEASUREMENT IN INDUSTRY

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1. INTRODUCTION

Microwave scattering from irregular or optically rough surfaces is often modelled as a scattering integral which is inherently difficult to solve. Most attempts at solution are either limited analytical methods which often fail to capture the essence of the problem or numerical simulations which address the whole problem but present difficulty in their physical interpretation.
In this paper we present a useful analytic solution technique based on the well-known Kirchhoff integral of physical optics. This modified method will be called Beckmann scattering theory, due to its extensive development by Beckmann & Spizzichino, [2], in electromagnetic scattering theory. This theory has two great advantages over other currently used methods such as integral equation methods and spectral techniques. It has simple physical intuition and second it permits non-trivial analytic solutions. The theory is applicable to the scattering of electromagnetic waves from irregular smooth surfaces whose mean radii of curvature at every scattering point is greater than the wavelength of of the incident wave. This seeming restriction is actually obeyed for a wide range of scattering problems. Currently, closed form solutions of the Beckmann integral may be achieved only for perfectly conducting periodic irregular surfaces. In reality, all materials are only approximately perfect conductors and rough surfaces are not necessarily periodic. The authors will present a solution of the Beckmann integral for periodic dielectric surfaces.
Another important application of Beckmann theory apart from its use in microwave scattering is in industry, particularly the non-contact temperature measurement of steel products. A commonly used technique in industrial temperature measurement is radiation pyrometry which measures the thermal radiation from an object's surface. In radiation pyrometry a vital parameter is the object's surface emissivity, ϵ, or its radiation ability as compared to a blackbody radiator. The accuracy of the temperature measurement is directly related to the emissivity which in turn may be determined by the rough surface scattering/reflection properties through Kirhhoff's law.
The surfaces of steel products at high temperatures in industry are often covered by layers of iron-oxide. Previous research has investigated the emissivity characteristics of optically flat unoxidised and (layered) oxidised steel surfaces, [1], [2]. This paper presents the results of an investigation into both unoxidised and oxidised rough surface scattering and its corresponding emissivity properties.
The theory has the added advantages of its applicability over a wide spectral range extending from the infrared (IR pyrometry) up to microwave (MW radiometry), and its ability to solve both the unoxidised and oxidised cases in a uniform manner.

2. BECKMANN THEORY

2.1 The Surface Roughness

The surface, S, is considered optically 'rough' in the sense described by Raleigh, [2].

2.1 The Beckmann Integral

A plane electromagnetic wave is incident upon an irregular surface given by the function $\zeta = \zeta(x)$ in the (x, z) plane. The ambient medium is air having electrical properties $\mu_0, \varepsilon_0, \sigma_0 = 0, N_0 = 1$ (permeability, permittivity, conductivity and refractive index) with the scattering surface having $\mu, \varepsilon, \sigma, N$. The linearly polarised (perpendicular ($\perp$) or parallel ($\|$), cf plane of incidence, (x, z) plane) wave is incident at a global angle θ_I and is scattered at a global observation angle θ_S, measured in the sense shown in Fig1.

Fig 1 Plane wave scattering at a rough interface

The local scattering angle is defined by $\vartheta = \theta_I - \beta$, where $\beta = \tan^{-1}\zeta'(x)$. The scattered field is evaluated in the Fraunhofer zone in terms of the scattering/reflection coefficient defined by $R_S = \rho_S R_0$, where ρ_S is the scattering coefficient and R_0 the reflection coefficient of a flat surface ($\zeta = 0$). The far field approximation imposes the restrictions $L/r \ll 1, kL^2/r \ll 1$ where measurements are taken at r with the dimensions of the scattering object given by L. The Beckmann integral is given by

$$R_S = -\frac{1}{4L\cos\theta_I}\int_{-L}^{L}(a\zeta' - b)e^{i(k_x x + k_z \zeta)}dx \qquad (1)$$

where $k_x = k(\sin\theta_I - \sin\theta_S)$, $k_z = -k(\cos\theta_I + \cos\theta_S)$ and $a = (1 - R)\sin\theta_I + (1 + R)\sin\theta_S$, $b = (1 + R)\cos\theta_S - (1 - R)\cos\theta_I$ where $R = R(\vartheta)$ is the local reflection coefficient.

2.2 The Tangent Plane Approximation

Beckmann theory is based on the Kirchhoff or Physical Optics approach to scattering and relies upon its range of validity on the so called 'Tangent Plane Approximation'. As long as the surface irregularities possess a large radius of curvature, r_c, compared to the incident wavelength, λ, of radiation, then the scattered field will be the field that would be present on the tangent plane at that scattering point. This is expressed mathematically as

$$4\pi r_c \cos\vartheta \gg \lambda \qquad (2)$$

3. ANALYTIC SOLUTIONS

3.1 Periodic Rough Surfaces

The surface S is assumed to be a periodic function, ie

$$\zeta(x) = \zeta(x + nT) = A\sin\left(\frac{2\pi x}{T}\right) \qquad (3)$$

of period T and amplitude A, n is an integer. Similarly all derivatives of ζ are also periodic. Let the size, L, of the scatterer be written in terms of integer multiples of the period, ie $L = nT$. Make all scattering observations at normal incidence in the specular direction, $\theta_I = \theta_S = 0$.

3.2 Varying Surface Reflection Coefficient

Following [3], define the local surface reflection coefficient, $R(\vartheta)$, in terms of a Taylor series expanded about the flat surface reflection coefficient, R_0 ie

$$R(\zeta') = R_0^{(0)} + R_0^{(1)}\zeta' + R_0^{(2)}\frac{\zeta'^2}{2!} + ... + O(\zeta'^m) + ... \qquad (4)$$

where $d^m R/d\zeta^m|_{\zeta'=0} = R_0^{(m)}$ for m an integer.

3.3 Solution for Sinusoidal Profiles

The integral (1) may be evaluated using term by term integration and the binomial theorem in terms of Bessel functions as

$$R_S = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\left(\frac{2\pi A}{T}\right)^{2m}\frac{R_0^{(2m)}}{2^{2m}(2m)!}\begin{pmatrix} 2m \\ 2j \end{pmatrix}\times$$
$$J_{(2m-2j)}\left(\frac{4\pi A}{\lambda}\right) \qquad (5)$$

which is the specular, analytic solution of the Beckmann integral at normal incidence for a sinusoidal rough surface in the range of validity $T \gg \sqrt{\pi A\lambda}$, by the tangent plane criterion.

3.4 Series Convergence

The evaluation of the integral is possible provided the Taylor series (4) is uniformly convergent. That is, if

$$\lim_{m\to\infty}\left|\frac{R_0^{(m+1)}}{R_0^{(m)}}\right|\frac{|\zeta'|}{m+1} < 1 \qquad (6)$$

and the result is independent of x and the m^{th} term approaches zero as $m \to \infty$. It can be shown that the series is uniformly convergent for all x and both types of polarisation, [2].

4. REAL SURFACES

4.1 Pure Surfaces

Purely rough surfaces, such as unoxidised steel surfaces in industry, can be approximated by periodic undulations as above. The term, $|2\pi A/T|$ is always small ($\ll 1$) except at small wavelengths and large periods at low temperatures ($< 200^o$C), [2]. Temperature measurement takes place at relatively high temperatures such as $700 - 1300^o$C. Then the emissivity of unoxidised steel surfaces may be approximated up to first order accuracy in the derivatives, R_0^m as

$$\epsilon = 1 - \left[J_0 \left(\frac{4\pi A}{\lambda} \right) \right]^2 R_{y0} \tag{7}$$

by Kirchhoff's law ($\epsilon = 1 - R_y$) and the definition of reflectivity ($R_y = R_S R_S^*$). The average reflectivity is defined for both kinds of polarisation $R_{y0} = (R_0^{\|} R_0^{*\|} + R_0^{\perp} R_0^{*\|})/2$, here "$*$" refers to complex conjugate.

4.2 Coated Surfaces

Coated surfaces such as oxidised steel surfaces may also be approximated in a mathematically equivalent manner. The emissivity of an oxidised steel surfaces with one layer of iron-oxide (FeO) with identical air/oxide-oxide/steel interface geometries can be written as

$$\epsilon_f = 1 - \left[J_0 \left(\frac{4\pi A}{\lambda} \right) \right]^2 R_{y_{f0}} \tag{8}$$

the reflectivity defined as above and

$$R_{f_0} = \frac{R_{1_0} + R_{2_0} e^{i\delta_0}}{1 + R_{1_0} R_{2_0} e^{i\delta_0}} \tag{9}$$

R_{j_0} are the flat surface reflection coefficients at each respective interface, measuring top ($_1$) to bottom ($_2$), and δ_0 representing the flat surface thin film interference effects.

5. RESULTS AND DISCUSSION

The integral, (1), may be solved numerically using a commercially available NAG routine specially tailored for the solution of this type of oscillatory integral.

5.1 Comparison of Theoretical and Numerical Calculations

A typical graph of ϵ verses λ was drawn for both the first order theoretical approximation, (7), and the full, numerically calculated integral, (1) at a roughness $A = 0.5\mu$m, $T = 100\mu$m at $T^o = 1000^o$C, see Fig 2. As can be seen the two graphs are identical. This implies that the first order approximation is a very accurate representation of the true solution over this wavelength range.

Fig. 2 Comparison of first order analytic and numerical solution of (7) for the emissivity over a range of wavelengths.

5.2 Comparison of Rough Surfaces with Optically Flat Layered Surfaces

Notice that the general trend of the graph shown in Fig 2 is very similar to the curves produced by optically flat layered surfaces studied previously, [1]. This suggests that the rough surface emissivity may be approximated by optically flat layered models. The following section investigates this possibility.

<u>Unoxidised Rough Surface</u>

Fig 3. is a graph of a pure unoxidised, rough steel surface with $A = 0.1\mu$m, $T = 100\mu$m, conductivity of steel, σ_{steel} calculated at 1000^oC, and a graph of an optically flat layer of iron-oxide (FeO), thickness $d_{layer} \simeq 1\frac{1}{2} \times A$, conductivity of σ_{layer} equivalent to FeO at about 1100^oC, on top of a steel substrate.

Fig 3. Graph of a slightly rough unoxidised steel surface and an optically flat layer of oxide on top of a steel substrate.

Fig 4 shows an emissivity graph of a rough surface with $A = 1\mu$m ($T = 100\mu$m, $\sigma_{steel}(1000^o)$C) and a graph of a single (optically flat) layer of FeO of thickness $d_{layer} \simeq 1\frac{1}{2} \times A$ and a conductivity $\sigma_{layer}(FeO$ at 900^oC).

Fig 4. Comparison between a rough unoxidised steel surface and an optically flat layer of oxide on steel.

Both Figures 3 and 4 show that it is possible to draw emissivity graphs of optically flat layered oxide surfaces which show the same general trends as emissivity graphs of pure unoxidised rough steel surfaces.

Oxidised Rough Surface

Fig 5 shows an emissivity graph of an oxidised rough steel surface with identical interface geometries and an oxide (FeO) thickness $d_{oxide} = 10\mu$m and a roughness $A = 1\mu$m ($T = 100\mu$m, both σ_{steel} and σ_{oxide} at 1000^oC). Also shown is a graph of two optically flat layers of oxide on top of a steel substrate with the first (bottom) layer that of the oxide, FeO, with the same thickness, $d_{oxide} = 10\mu$m and conductivity as the original rough oxidised surface, the second (topmost) is an extra layer of oxide (FeO) of thickness, $d_{layer} = 1\frac{1}{2} \times A$ and conductivity $\sigma_{layer}(FeO$ at 900^o)C.

Fig 5. Two-layer, (optically) flat approximation to a rough oxidised surface having identical interface geometries

As for the unoxidised case, it is possible to approximate the rough oxidised surface as a set of two optically flat layers of oxide on top of a steel substrate. In each case the roughness characteristics may be approximated by making a suitable choice of layer thickness and conductivity.

6. CONCLUSIONS

Using a series approximation to the reflection coefficient at the local scattering point, a closed form solution of the Beckmann scattering integral was derived. This analytic solution led to the following conclusions :

- The solution, (5), provides a simple, accurate, physically intuitive result in terms of a series of integer order Bessel functions and equivalent flat surface reflection coefficients depending critically on the roughness parameter $4\pi A/\lambda$. The series is uniformly convergent for all values of x in the range of integration.

- The comparisons above suggest that unoxidised and oxidised steel surfaces may be approximated by optically flat, thin films of oxide on top of a steel substrate. In each case the roughness, A, of the surface may be approximated by a thin (topmost) layer of material comprised of iron-oxide, FeO, of thickness $d_{layer} \simeq 1\frac{1}{2} \times A$ and a conductivity, σ_{layer} of FeO in the temperature range $900 - 1100^o$C. For the oxidised case the bottom layer of material is made of the original oxide having the same thickness and conductivity as the original.

- Because of the simplicity of flat layer models it is of great value to further investigate the relationship between flat and rough surface emissivity models.

7. REFERENCES

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